Topology and its Applications 190 (2015) 42–58

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

# Products of bounded subsets of paratopological groups

Iván Sánchez *, Mikhail G. Tkachenko [1]

*Departamento de Matemáticas, Universidad Autónoma Metropolitana, Av. San Rafael Atlixco 186, Col. Vicentina, Del. Iztapalapa, C.P. 09340, D.F., Mexico*

ARTICLE INFO

*Article history:*
Received 4 February 2014
Received in revised form 19 July 2014
Accepted 11 March 2015
Available online xxxx

*MSC:*
primary 22A05, 54H11
secondary 54D30, 54G20

*Keywords:*
Compact
Precompact
Bounded
Strongly bounded
Hausdorff number
Totally $\omega$-narrow

ABSTRACT

We prove that if $B_i$ is a bounded subset of a totally $\omega$-narrow paratopological group $G_i$, where $i \in I$, then $\prod_{i\in I} B_i$ is bounded in $\prod_{i\in I} G_i$. The same conclusion remains valid in the case of products of bounded subsets of Hausdorff commutative paratopological groups with countable Hausdorff number or products of Lindelöf paratopological groups. In fact, we show that if $B$ is a bounded subset of a paratopological group $G$ satisfying one of the conditions (a)–(d) below, then $B$ is strongly bounded in $G$:

(a) $G$ is totally $\omega$-narrow;
(b) $G$ is commutative, Hausdorff, and has countable Hausdorff number;
(c) $G$ is saturated and weakly Lindelöf;
(d) $G$ is Lindelöf.

These results imply that boundedness of subsets is productive in the classes of paratopological groups listed in (a)–(d).


## 1. Introduction

The product of an arbitrary family of pseudocompact topological groups is pseudocompact—this is the celebrated theorem proved by Comfort and Ross in [7]. Recently, A. Ravsky [13] proved a similar result for products of pseudocompact paratopological groups. To be more accurate, we have to reformulate Ravsky's theorem as follows: *An arbitrary product of feebly compact paratopological groups is feebly compact.* As usual, a space $X$ is called *feebly compact* if every locally finite family of open sets in $X$ is finite. Feebly compact spaces are not assumed to satisfy any separation axiom, while pseudocompact spaces are necessarily

* Corresponding author.
*E-mail addresses:* isr.uami@gmail.com (I. Sánchez), mich@xanum.uam.mx (M.G. Tkachenko).
[1] The second-listed author was partially supported by the Consejo Nacional de Ciencia y Tecnología (CONACyT) of Mexico, grant number CB-2012-01 178103.

http://dx.doi.org/10.1016/j.topol.2015.03.017

Tychonoff. It is clear that feeble compactness and pseudocompactness coincide in Tychonoff spaces, so the former notion is a 'right' extension of the latter one to non-Tychonoff spaces.

The Comfort–Ross theorem on the productivity of pseudocompactness in topological groups admits several extensions to wider classes of spaces (see [1,6,31,32]). Another generalization was obtained by the second-listed author in [22, Theorem 2.2]: If $B_i$ is a bounded subset of a topological group $G_i$, for each $i \in I$, then $\prod_{i\in I} B_i$ is bounded in $\prod_{i\in I} G_i$.

Let us recall that a subset $B$ of a space $X$ is said to be *bounded* in $X$ if every locally finite family of open sets in $X$ contains only finitely many elements that meet $B$. Hence boundedness is a *relative* version of feeble compactness. It is clear from the definition that a subset $B$ of a *Tychonoff* space $X$ is bounded in $X$ if and only if every continuous real-valued function defined on $X$ is bounded on $B$.

With Ravsky's theorem in mind, it is natural to ask whether boundedness remains productive for subsets of paratopological groups (see [19, Problem 2.16] and [26, Problem 7.1]). We prove in Theorem 2.8 that this is indeed the case if the paratopological groups are additionally assumed to be *totally ω-narrow* (the factors in the theorem are not assumed to satisfy any separation axiom). The same conclusion is valid for products of bounded subsets of Hausdorff commutative paratopological groups with countable Hausdorff number (Corollary 3.10) and for products of bounded subsets of saturated, weakly Lindelöf paratopological groups (Corollary 3.14). The latter fact implies that a similar conclusion holds for products of bounded subsets of precompact paratopological groups (see Corollary 3.16).

The key technical notion in this article is the one called *strong boundedness* (see Definition 2.3). It is known that every bounded subset of a topological group is strongly bounded in the group [24, Theorem 2]. We show that in all the aforementioned cases, a bounded subset of a paratopological group turns out to be strongly bounded in the group. Then we apply the fact that strong boundedness is productive for subsets of topological spaces [22, Theorem 2.6]. The question of whether a bounded subset $B$ of an arbitrary paratopological group $G$ is strongly bounded in $G$ remains open, even if $B$ is countably compact (see Problem 5.1).

Section 5 of the article contains several open problems on bounded sets in paratopological groups which are supplied with brief comments.

### *1.1. Notation, terminology, and preliminary facts*

A *paratopological* group is a group with a topology such that multiplication on the group is jointly continuous. The wording *an isomorphism of paratopological groups* does not necessarily mean that the isomorphism in question is continuous.

If $\tau$ is the topology of a paratopological group $G$, then the family

$$\tau^{-1} = \{U^{-1} : U \in \tau\}$$

is also a topology on $G$ and $G' = (G, \tau^{-1})$ is again a paratopological group *conjugated* to $G$. It is clear that the inversion on $G$ is a homeomorphism of $G$ onto $G'$. The upper bound $\tau^* = \tau \vee \tau^{-1}$ is a topological group topology on $G$ and $G^* = (G, \tau^*)$ is a topological group *associated* to $G$.

A paratopological (topological) group $G$ is said to be *ω-narrow* if for every neighborhood $U$ of the neutral element in $G$, there exists a countable set $C \subseteq G$ such that $CU = G = UC$. We call a paratopological group $G$ *totally ω-narrow* if the topological group $G^*$ associated to $G$ is $\omega$-narrow. In topological groups, total $\omega$-narrowness and $\omega$-narrowness coincide, but the Sorgenfrey line is an example of an $\omega$-narrow (even Lindelöf) paratopological group which fails to be totally $\omega$-narrow.

A paratopological group $G$ is *ω-balanced* if for every neighborhood $U$ of the identity $e$ in $G$, one can find a countable family $\gamma$ of open neighborhoods of $e$ in $G$ such that for every $x \in G$, there exists $V \in \gamma$ with $xVx^{-1} \subseteq U$. It is clear that every paratopological Abelian group is $\omega$-balanced.

A subset $U$ of a space $X$ is called *regular open* if $U = \operatorname{Int} \overline{U}$. Given a space $(X, \tau)$, denote by $\tau'$ the topology on $X$ whose base consists of regular open subsets of $(X, \tau)$. The space $(X, \tau')$ is said to be the *semiregularization* of $(X, \tau)$ and is denoted by $X_{sr}$. It is easy to see that $\tau' \subseteq \tau$ and that the spaces $(X, \tau)$ and $(X, \tau')$ have the same regular open subsets. A space whose regular open subsets form a base for its topology is called *semiregular.*

Answering a question posed by the authors in a previous version of this article, T. Banakh and A. Ravsky [4] established the following useful fact.

**Theorem 1.1.** *Let $N$ be an arbitrary subgroup of a paratopological group $G$. Then the semiregularization $(G/N)_{sr}$ of the quotient space $G/N$ satisfies the $T_3$ separation axiom. If therefore $G/N$ is Hausdorff, then $(G/N)_{sr}$ is a regular space.*

The operation of semiregularization has another useful property mentioned in [13, Lemma 20]:

**Lemma 1.2.** *Let $\{X_i : i \in I\}$ be a family of spaces and $X = \prod_{i \in I} X_i$ the product of this family. Then the semiregularization of $X$ is naturally homeomorphic to the product space $\prod_{i \in I} (X_i)_{sr}$.*

Another useful operation in the category of paratopological groups is the so-called *topological group reflection*. Given a paratopological group $G$, the topological group reflection of $G$, denoted by $G_*$, is the group $G$ endowed with the finest topological group topology weaker than the original topology of $G$. A description of the topology of $G_*$ in terms of the group $G$ is given in [25]. It is easy to verify that the operation of taking the topological group reflection is a *covariant functor* from the category of paratopological groups to the category of topological groups. This means, in particular, that for every continuous homomorphism $f: G \to H$ of paratopological groups, the corresponding homomorphism $f_*: G_* \to H_*$ coinciding with $f$ pointwise is also continuous.

## 2. Bounded sets in totally $\omega$-narrow paratopological groups

Let $B$ be a subset of a space $X$. We say that $X$ is *regular on* $B$ if for every point $x \in B$ and every closed subset $F$ of $B$ with $x \notin F$, the sets $\{x\}$ and $F$ have disjoint open neighborhoods in $X$. It is clear that a regular space $X$ is regular on *every* set $B \subseteq X$.

**Lemma 2.1.** *Let $f: X \to Y$ be a continuous one-to-one mapping and $B$ a bounded subset of $X$. Suppose also that $X$ is regular on $B$ and every point of $Y$ is the intersection of countably many of its closed neighborhoods. Then the restriction of $f$ to $B$ is a homeomorphism.*

**Proof.** Suppose for a contradiction that the restriction of $f$ to $B$ is not a homeomorphism when considered as a mapping of $B$ onto $f(B)$. Then we can find a point $x \in B$ and an open neighborhood $U$ of $x$ in $X$ such that $(B \cap f^{-1}(O)) \setminus U \neq \emptyset$, for every neighborhood $O$ of the point $y = f(x)$ in $Y$. Then $F = B \setminus U$ is a closed subset of $B$ and $x \notin F$, so there exist disjoint open sets $U_x$ and $U_F$ in $X$ such that $x \in U_x$ and $F \subseteq U_F$. Then the open neighborhood $V = U \cap U_x$ of $x$ in $X$ satisfies $\overline{V} \cap B \subseteq U$.

It follows from our assumption about $Y$ that there exists a decreasing sequence $\{O_n : n \in \omega\}$ of open neighborhoods of $y$ in $Y$ such that $\{y\} = \bigcap_{n \in \omega} \overline{O_n}$. Then $W_n = f^{-1}(O_n) \setminus \overline{V}$ is an open subset of $X$ and our choice of $x \in B$ and the sets $U$ and $V$ implies that $W_n \cap B \neq \emptyset$, for each $n \in \omega$. We claim that the family $\{W_n : n \in \omega\}$ is locally finite in $X$. Indeed, since $f$ is continuous and one-to-one, it follows from $f(W_n) \subseteq O_n$ for $n \in \omega$ that the family $\{W_n : n \in \omega\}$ can accumulate only at the point $x$. Since $W_n \cap V = \emptyset$ for each $n \in \omega$, our claim follows.

Each element of the infinite locally finite family $\{W_n : n \in \omega\}$ meets $B$, so $B$ fails to be bounded. This contradiction completes the proof. □

It is worth noting that in Lemma 2.1, the condition '$f$ is one-to-one' can be weakened to 'the restriction of $f$ to $B$ is one-to-one'.

**Corollary 2.2.** *Let $f: X \to Y$ be a continuous one-to-one mapping of a regular space $X$ to a space $Y$. Suppose also that every point of $Y$ is the intersection of countably many of its closed neighborhoods. Then:*

(a) *the restriction of $f$ to every bounded subset of $X$ is a homeomorphism;*
(b) *the image $f(B)$ of a closed bounded subset $B$ of $X$ is closed in $Y$.*

**Proof.** The claim in (a) follows directly from Lemma 2.1, so it suffices to verify (b). Suppose that $B$ is a closed bounded subset of $X$. If $y \in Y \setminus f(B)$, take a point $x \in X$ such that $f(x) = y$. Then $B' = B \cup \{x\}$ is a bounded subset of $X$ and $x$ is isolated in $B'$, so (a) implies that $y = f(x)$ is isolated in $f(B) \cup \{y\}$. It follows that $f(B)$ is closed in $Y$. □

**Definition 2.3.** A subset $B$ of a space $X$ is said to be strongly bounded in $X$ if every infinite family of open sets in $X$ each of which meets $B$, contains an infinite subfamily $\{U_n : n \in \omega\}$ satisfying the following property:

(∗) For every filter $\mathcal{F}$ of infinite subsets of $\omega$, the set $\bigcap_{F \in \mathcal{F}} \overline{\bigcup_{n \in F} U_n}$ is non-empty.

It is clear that every strongly bounded subset of $X$ is bounded in $X$, but the converse is false (see [10]). The proof of the following lemma is straightforward, but we present it here for the sake of completeness.

**Lemma 2.4.** *Let $G_{sr}$ be the semiregularization of a paratopological group $G$. A subset $B$ of $G$ is bounded in $G$ iff $B$ is bounded in $G_{sr}$. Similarly, $B$ is strongly bounded in $G$ iff $B$ is strongly bounded in $G_{sr}$.*

**Proof.** It is clear that $B$ is bounded in $G_{sr}$, for each bounded subset $B$ of $G$. Suppose that $B \subseteq G$ is not bounded in $G$. Then there exists an infinite locally finite family $\{U_n : n \in \omega\}$ of open sets in $G$ such that $U_n \cap B \neq \emptyset$ for each $n \in \omega$. We claim that the family $\{\mathrm{Int}\,\overline{U_n} : n \in \omega\}$ of open sets in $G_{sr}$ is locally finite in $G_{sr}$. Indeed, for an arbitrary point $x \in G$, take an open neighborhood $V$ of $x$ which meets at most finitely many of $U_n$'s. Then the set $i(\mathrm{Int}\,\overline{V})$ is an open neighborhood of $x$ in $G_{sr}$ which meets only finitely many elements of the family $\{\mathrm{Int}\,\overline{U_n} : n \in \omega\}$. Since each open set $\mathrm{Int}\,\overline{U_n}$ meets $B$, the set $B$ is not bounded in $G_{sr}$. This proves the first claim of the lemma.

Let $B$ be a strongly bounded subset of $G_{sr}$. Given a family $\{U_n : n \in \omega\}$ of open subsets of $G$ meeting $B$, we define the sets $V_n = \mathrm{Int}\,\overline{U_n}$, for $n \in \omega$. Then each of the sets $V_n$'s is open in $G_{sr}$ and meets $B$ since $U_n \subseteq V_n$. Suppose that the family $\{V_n : n \in \omega\}$ satisfies condition (∗) of Definition 2.3. Hence there exists a point $x_0 \in G$ such that every neighborhood of $x_0$ in $G_{sr}$ meets infinitely many elements of the family $\{V_n : n \in F\}$, for each $F \in \mathcal{F}$. We claim that the same happens for every neighborhood of $x_0$ in $G$, i.e. the family $\{U_n : n \in \omega\}$ also satisfies condition (∗). Indeed, take a filter $\mathcal{F}$ of infinite subsets of $\omega$ and an element $F \in \mathcal{F}$. If $U$ is an open neighborhood of $x_0$ in $G$, let $V = \mathrm{Int}\,\overline{U}$. Then $V$ is an open neighborhood of $x_0$ in $G_{sr}$, so $V \cap V_n \neq \emptyset$ for infinitely many $n \in F$. It now follows from the definition of $V$ and $V_n$ that $U \cap U_n \neq \emptyset$ for each such an element $n \in F$. Hence $B$ is strongly bounded in $G$. The converse implication is evident. □

The following fact and its proof are taken from [33].

**Lemma 2.5.** *Let $f: X \to Y$ be a continuous mapping of $X$ to a $T_3$-space $Y$. Then $f$ remains continuous as a mapping of the semiregularization $X_{sr}$ of $X$ to $Y$.*

**Proof.** Take a point $x \in X$ and a neighborhood $U$ of $f(x)$ in $Y$. Let $V$ be an open neighborhood of $f(x)$ such that $\overline{V} \subseteq U$. Since $f$ is continuous on $X$, we can find an open neighborhood $O$ of $x$ in $X$ such that $f(O) \subseteq V$. Hence $f(\overline{O}) \subseteq \overline{V} \subseteq U$ and, consequently, $f(\operatorname{Int}\overline{O}) \subseteq f(\overline{O}) \subseteq U$. Since $\operatorname{Int}\overline{O}$ is an open neighborhood of $x$ in $X_{sr}$, we see that $f$ is continuous on $X_{sr}$. □

The following fact complements Lemma 2.5.

**Lemma 2.6.** *Let $f: X \to Y$ be a continuous d-open mapping. Then $f$ remains continuous as a mapping of $X_{sr}$ to $Y_{sr}$.*

**Proof.** Take an arbitrary point $x \in X_{sr}$ and an open neighborhood $U$ of $y = f(x)$ in $Y_{sr}$. We can assume without loss of generality that $U$ is a regular open set in the space $Y$. By the continuity of the mapping $f: X \to Y$, there exists an open neighborhood $V$ of $x$ in $X$ such that $f(V) \subseteq U$. Since $V \subseteq \operatorname{Int}\overline{V}$ (the interior and closure are taken in $X$), the set $V_0 = \operatorname{Int}\overline{V}$ is an open neighborhood of $x$ in $X_{sr}$. Clearly, $V_0$ is open in $X$ and $V$ is dense in $V_0$.

Since the mapping $f: X \to Y$ is $d$-open, there exists an open set $W$ in $Y$ which contains $f(V_0)$ as a dense subset. Using the continuity of $f$, we obtain that

$$f(V_0) \subseteq W \subseteq \overline{W} = \overline{f(V_0)} = \overline{f(V)} \subseteq \overline{U},$$

where the closures and interior are taken in $Y$. Since $W$ is open in $Y$, the above inclusions imply that $f(V_0) \subseteq \operatorname{Int}\overline{U} = U$. This proves the continuity of the mapping $f: X_{sr} \to Y_{sr}$. □

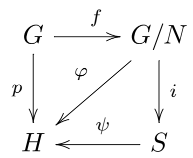
**Theorem 2.7.** *Every bounded subset of a totally $\omega$-narrow paratopological group $G$ is strongly bounded.*

**Proof.** It follows from Corollary 2.10 of [29] that, for a subset $A$ of an arbitrary paratopological group $H$, the set $\varphi_{H,r}(A)$ is bounded (strongly bounded) in the regular reflection $Reg(H)$ of $H$ if and only if $A$ is bounded (strongly bounded) in $H$, where $\varphi_{H,r}$ is the canonical homomorphism of $H$ onto $Reg(H)$. It is also clear that $Reg(H)$ is totally $\omega$-narrow provided so is $H$. Therefore, it suffices to prove the theorem in the case of a regular paratopological group.

Let $B$ be a bounded subset of a regular totally $\omega$-narrow paratopological group $G$. Suppose that $\gamma = \{U_n : n \in \omega\}$ is an infinite family of open sets in $G$ such that $B \cap U_n \neq \emptyset$, for each $n \in \omega$. We claim that $\gamma$ satisfies condition $(*)$ of Definition 2.3. Since the group $G$ is regular and totally $\omega$-narrow, it embeds as a subgroup into a product of regular second countable paratopological groups [14, Corollary 2.4]. Equivalently, there exists a family $\mathcal{L}$ of continuous homomorphisms of $G$ onto regular second countable paratopological groups which generates the original topology of $G$. Therefore, we can additionally assume that for every $n \in \omega$, the set $U_n$ has the form $U_n = p_n^{-1}(\widetilde{U}_n)$, where $p_n \in \mathcal{L}$ and $\widetilde{U}_n$ is an open subset of the regular second countable paratopological group $p_n(G)$. Let $p$ be the diagonal product of the family $\{p_n : n \in \omega\}$ and $H = p(G)$. Then $p$ is a continuous homomorphism of $G$ onto the regular second countable paratopological group $H$, a subgroup of the product $\prod_{n\in\omega} p_n(G)$. It follows from the definition of $p$ that for every $n \in \omega$, there exists an open subset $W_n$ of $H$ satisfying $U_n = p^{-1}(W_n)$.

Let $N$ be the kernel of the homomorphism $p$ and $K = G/N$ the corresponding quotient group. Denote by $f$ the quotient homomorphism of $G$ onto $G/N$. Clearly there exists a continuous isomorphism (not necessarily a homeomorphism) $\varphi$ of $G/N$ onto $H$ satisfying $p = \varphi \circ f$. Notice that the group $G/N$ is Hausdorff since $\varphi$ is a continuous bijection and $H$ is regular.

Denote by $S$ the semiregularization of the group $G/N$ and let $i$ be the identity mapping of $G/N$ onto $S$. Then $S$ is a regular space by Theorem 1.1. Let $\psi: S \to H$ be a bijection satisfying $\varphi = \psi \circ i$. As $\varphi$ is continuous, $H$ is regular, and $S$ is the semiregularization of $G/N$, it follows from Lemma 2.5 that $\psi$ is also continuous.




The sets $f(B)$, $i(f(B))$, and $p(B)$ are bounded in $G/N$, $S$, and $H$, respectively. The closure of $i(f(B))$ in $S$, say, $C$ is also bounded, so Corollary 2.2 implies that $\psi(C)$ is a closed bounded subset of $H$. As $H$ is regular and second countable, the set $\psi(C)$ is compact. Applying Corollary 2.2 once again, we conclude that the restriction of $\psi$ to $C$ is a homeomorphism of $C$ onto $\psi(C)$, so the set $C$ is compact as well.

For every $n \in \omega$, let $O_n = \psi^{-1}(W_n)$. Since $O_n \cap C \neq \emptyset$ for each $n \in \omega$, we conclude that

$$C \cap \bigcap_{F \in \mathcal{F}} \overline{\bigcup_{n \in F} O_n} \neq \emptyset,$$

where $\mathcal{F}$ is a filter of infinite subsets of $\omega$. Pick an arbitrary point $z$ in the above intersection and take points $y \in G/N$ and $x \in G$ with $i(y) = z$ and $f(x) = y$. We claim that $x \in \bigcap_{F \in \mathcal{F}} \overline{\bigcup_{n \in F} U_n}$. Indeed, since $S$ is the semiregularization of the Hausdorff space $G/N$, the mapping $i$ establishes a one-to-one correspondence between the regular closed sets in $G/N$ and $S$. In other words, $\overline{i^{-1}(O)} = i^{-1}(\overline{O})$ is a regular closed subset of $S$, for every open subset $O$ of $S$. Therefore, if $V_n = i^{-1}(O_n)$ for $n \in \omega$, then $y \in \overline{\bigcup_{n \in F} V_n}$, for each $F \in \mathcal{F}$. Since the mapping $f$ is open and continuous and $f(x) = y$, we have that

$$x \in f^{-1}\left(\overline{\bigcup_{n \in F} V_n}\right) = \overline{f^{-1}\left(\bigcup_{n \in F} V_n\right)} = \overline{p^{-1}\left(\bigcup_{n \in F} W_n\right)} = \overline{\bigcup_{n \in F} U_n},$$

for each $F \in \mathcal{F}$. It follows that $\bigcap_{F \in \mathcal{F}} \overline{\bigcup_{n \in F} U_n} \neq \emptyset$, so the family $\{U_n : n \in \omega\}$ satisfies $(*)$. We have thus proved that $B$ is strongly bounded in $G$. □

According to [22, Theorem 2.6], if $B_i$ is a strongly bounded subset of a space $X_i$, where $i \in I$, then the product of $\prod_{i \in I} B_i$ is strongly bounded in $\prod_{i \in I} X_i$. Combining this result with Theorem 2.7, we obtain the following:

**Theorem 2.8.** *Let $B_i$ be a bounded subset of a totally $\omega$-narrow paratopological group $G_i$, for each $i \in I$. Then $\prod_{i \in I} B_i$ is bounded in $\prod_{i \in I} G_i$.*

**Corollary 2.9.** *If $A$ and $B$ are bounded subsets of a totally $\omega$-narrow paratopological group $G$, then the group product $AB$ of the sets $A$ and $B$ is bounded in $G$.*

**Proof.** Let us note that the subset $AB$ of $G$ is the image of the set $A \times B$ under the continuous mapping $m$ of $G \times G$ to $G$ defined by $m(x, y) = xy$. Since the product $A \times B$ is bounded in $G \times G$ by Theorem 2.8, the set $AB$ is bounded in $G$. □

It is also known that if $B_1$ is a strongly bounded subset of a space $X_1$ and $B_2$ is a bounded subset of a space $X_2$, then $B_1 \times B_2$ is bounded in $X_1 \times X_2$ [22, Corollary 2.5]. This fact along with Theorem 2.7 implies the next result:

**Corollary 2.10.** *If $B$ is a bounded subset of a totally $\omega$-narrow paratopological group $G$ and $C$ is a bounded subset of a space $Y$, then $B \times C$ is bounded in $G \times Y$.*

In the special case when a bounded subset $B$ of a paratopological group $G$ coincides with the whole of $G$ (i.e. when $G$ is feebly compact), one can drop the assumption of total $\omega$-narrowness of the group $G$ in Corollary 2.10:

**Corollary 2.11.** *If $G$ is a feebly compact paratopological group and $C$ is a bounded subset of a space $Y$, then $G \times C$ is bounded in $G \times Y$.*

**Proof.** Let $P = G \times Y$. Then, according to Lemma 1.2, the semiregularization of $P$ is naturally homeomorphic to the product $G_{sr} \times Y_{sr}$ of the semiregularizations of $G$ and $Y$. By Lemma 2.4, it suffices to show that the set $G \times C$ is bounded in $G_{sr} \times Y_{sr}$.

Since $G$ is feebly compact, it follows from [13, Lemma 9] that $G_{sr}$ is a (not necessarily Hausdorff) topological group. Clearly $G_{sr}$ is a continuous image of $G$, so the topological group $G_{sr}$ is also feebly compact, i.e. $G_{sr}$ is bounded in itself. Since $C$ is bounded in $Y_{sr}$, [10, Theorem 1.2] implies that $G \times C$ is bounded in $G \times Y$. □

## 3. Paratopological groups with countable Hausdorff number

Let us say that a paratopological group $G$ has property (wH) if

$$\bigcap_{U \in \mathcal{N}(e)} \overline{U} = \bigcap \mathcal{N}(e).$$

As usual, $\mathcal{N}(e)$ stands for the family of open neighborhoods of the neutral element $e$ in $G$. It is clear that $G$ satisfies condition (wH) provided it is either Hausdorff or a $T_3$-space. The following definition is slightly more general than the corresponding one in [23], where the Hausdorff number was defined only for Hausdorff paratopological groups.

**Definition 3.1.** Let $G$ be a paratopological group with property (wH). The *Hausdorff number* of $G$ is the minimum cardinal number $\kappa \geq 1$ such that for every neighborhood $U$ of the neutral element $e$ in $G$, there exists a family $\gamma$ of open neighborhoods of $e$ in $G$ such that $\bigcap_{V \in \gamma} VV^{-1} \subseteq U$ and $|\gamma| \leq \kappa$.

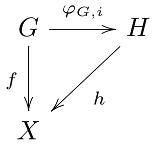
It follows from the above definition that a paratopological group $G$ satisfying $Hs(G) = 1$ is a topological group.

Every Lindelöf paratopological group satisfying the Hausdorff separation axiom has countable Hausdorff number [23, Proposition 2.4]. A direct verification shows that the same conclusion remains valid for Lindelöf paratopological groups with property (wH). It is also easy to see that the class of paratopological groups with countable Hausdorff number is productive [23, Proposition 2.3].

In this section we deduce from more general results that every bounded subset of a paratopological abelian group with countable Hausdorff number is strongly bounded (see Theorem 3.9). Therefore, bounded subsets of the groups from this class are stable with respect to taking arbitrary products (Corollary 3.10).

We start with several auxiliary results and known facts explaining our approach.

Given an arbitrary paratopological group $G$ and a number $i \in \{0, 1, 2, 3, 3.5\}$, a *$T_i$-reflection* of $G$, denoted by $T_i(G)$, is a pair $(H, \varphi_{G,i})$, where $H$ is a paratopological group satisfying the $T_i$ separation axiom and $\varphi_{G,i}$ is a continuous homomorphism of $G$ onto $H$ with the following property: *For every continuous mapping $f: G \to X$ to a $T_i$-space $X$, there exists a continuous mapping $h: H \to X$ such that $f = h \circ \varphi_{G,i}$.*




Similarly, a *regular reflection* of a paratopological group $G$ is a pair $(H, \varphi_{G,r})$, where $H$ is a regular paratopological group and $\varphi_{G,r}$ is a continuous homomorphism of $G$ onto $H$ such that every continuous mapping of $G$ to a regular space admits a continuous factorization through $\varphi_{G,r}$. The corresponding group $H$ is denoted by $Reg(G)$. By 'regular' we mean '$T_1$ & $T_3$'.

Abusing of terminology, we will usually refer to $T_2(G)$ and $Reg(G)$ as the Hausdorff and regular reflection, respectively, of the group $G$. It is shown in Propositions 2.2 and 2.5 of [27] that for every paratopological group $G$, the Hausdorff and regular reflections of $G$ exist and are unique up to a topological isomorphism.

In fact, these results are proved in [27] for the wider class of *semitopological* groups, so $T_2(G)$ and $Reg(G)$ are formally known to be semitopological groups only. However, the functors $T_2$ and $Reg$ preserve paratopological groups as well, as mentioned on page 374 of [27] or in the introduction of [28].

The following fact explains our interest in the Hausdorff and regular reflections of paratopological groups (see Proposition 2.9 and Corollary 2.10 of [29]):

**Proposition 3.2.** *Let $G$ be an arbitrary paratopological group. The following conditions are equivalent for a subset $B$ of $G$:*

(a) *$B$ is (strongly) bounded in $G$;*
(b) *$\varphi_{G,2}(B)$ is (strongly) bounded in $T_2(G)$;*
(c) *$\varphi_{G,r}(B)$ is (strongly) bounded in $Reg(G)$.*

Proposition 3.2 shows that one can study (strongly) bounded sets in *regular* paratopological groups exclusively.

Another useful and important fact is that the regular reflection of a paratopological group $G$ can be obtained from $G$ in two steps (see Corollary 2.8 of [28]):

**Proposition 3.3.** *For every paratopological group $G$, the regular reflection of $G$ is topologically isomorphic to the semiregularization of $T_2(G)$, i.e. $Reg(G) \cong (T_2(G))_{sr}$. In particular, $Reg(G)$ is an image of $T_2(G)$ under a continuous one-to-one homomorphism.*

A subgroup $H$ of a topological group $G$ is called *admissible* if there exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the identity in $G$ such that $U_{n+1}^3 \subseteq U_n$ for each $n \in \omega$ and $H = \bigcap_{n\in\omega} U_n$.

**Lemma 3.4.** *([2, Lemma 5.5.2]) Let $G$ be a topological group. Then:*

a) *every admissible subgroup $H$ of $G$ is closed in $G$;*
b) *every neighborhood of the identity in $G$ contains an admissible subgroup;*
c) *the intersection of countable many admissible subgroups of $G$ is again an admissible subgroup of $G$.*

According to [3], a paratopological group $G$ is 2-*oscillating* if for any neighborhood $U$ of the identity $e$ in $G$ there is a neighborhood $V$ of $e$ such that $V^{-1}V \subseteq UU^{-1}$. All precompact and all abelian paratopological groups are 2-oscillating (see [3, Proposition 3]).

**Lemma 3.5.** *Let $G$ be a 2-oscillating paratopological group with countable Hausdorff number. Then for every family $\{W_n : n \in \omega\} \subseteq \mathcal{N}(e)$, there exists a closed subgroup $H$ of $G_*$ such that $G_*/H$ is submetrizable and $H \subseteq \bigcap_{n\in\omega} W_n$.*

**Proof.** Let $\{W_n : n \in \omega\}$ be a countable family of open neighborhoods of the identity $e$ in $G$. Since $G$ has countable Hausdorff number, there exists a countable family $\gamma \subseteq \mathcal{N}(e)$ such that $\bigcap_{V\in\gamma} VV^{-1} \subseteq \bigcap_{n\in\omega} W_n$. It follows from [3, Theorem 1] that the set $VV^{-1}$ is open in $G_*$ for each $V \in \mathcal{N}(e)$. By Lemma 3.4, we can find an admissible subgroup $H$ of $G_*$ satisfying $H \subseteq \bigcap_{V\in\gamma} VV^{-1} \subseteq \bigcap_{n\in\omega} W_n$. Also, Lemma 3.4 implies that $H$ is closed in $G_*$. Since $H$ is an admissible subgroup of $G_*$, the quotient space $G_*/H$ is submetrizable (see [2, Lemma 6.10.7]). Hence $G/H$ is submetrizable as well. □

**Lemma 3.6.** *If $G$ is a 2-oscillating paratopological group, then so is $Reg(G)$.*

**Proof.** Put $H = G_{sr}$. Let us show that $H$ is 2-oscillating. Take an open neighborhood $U$ of the identity $e$ in $G$. Then $\operatorname{Int}\overline{U}$ is an open neighborhood of $e$ in $H$. Since $G$ is 2-oscillating, the set $WW^{-1}$ is open in $G_*$ for each open neighborhood of $e$ in $G$. So we can find an open neighborhood $V$ of $e$ in $G$ such that $(VV^{-1})^2 \subseteq (\operatorname{Int}\overline{U})(\operatorname{Int}\overline{U})^{-1}$. It is clear that $\overline{V} \subseteq VV^{-1}$, whence $\operatorname{Int}\overline{V} \subseteq VV^{-1}$. Hence $(\operatorname{Int}\overline{V})^{-1} \subseteq VV^{-1}$. We conclude therefore that

$$(\operatorname{Int}\overline{V})^{-1}(\operatorname{Int}\overline{V}) \subseteq (VV^{-1})^2 \subseteq (\operatorname{Int}\overline{U})(\operatorname{Int}\overline{U})^{-1}.$$

This proves that the group $H = G_{sr}$ is 2-oscillating.

According to [27, Theorem 3.8] and [28, Theorem 2.5], we have the equalities $Reg(G) = T_0(T_3(G))$ and $T_3(G) = G_{sr} = H$. There exists a continuous open homomorphism $\varphi_{H,0}$ of $H$ onto $T_0(H)$ (see [27, Proposition 2.5]). Since $H$ is 2-oscillating and $\varphi_{H,0}$ is a continuous open homomorphism, the paratopological group $T_0(H) = Reg(G)$ is 2-oscillating as well. □

**Theorem 3.7.** *Let $G$ be a 2-oscillating paratopological group satisfying the condition $Hs(Reg(G)) \le \omega$. Then all bounded subsets of $G$ are strongly bounded.*

**Proof.** It follows from Proposition 3.2 that there is a natural correspondence between bounded subsets of the groups $G$ and $Reg(G)$, and the same is valid for strongly bounded subsets of $G$ and $Reg(G)$. By Lemma 3.6, $Reg(G)$ is a 2-oscillating paratopological group. Therefore, we can assume without loss of generality that $G = Reg(G)$, i.e. that $G$ is regular.

Let $B$ be a bounded subset of $G$. Suppose that $\gamma = \{U_n : n \in \omega\}$ is an infinite family of open sets in $G$ such that $B \cap U_n \neq \emptyset$, for each $n \in \omega$.

Let us show that $\gamma$ satisfies condition $(*)$ of Definition 2.3. For every $n \in \omega$, we can find $x_n \in B \cap U_n$ and $W_n \in \mathcal{N}(e)$ such that $x_n W_n^2 \subseteq U_n$. By Lemma 3.5, there exists a closed subgroup $N$ of $G_*$ such that $G_*/N$ is submetrizable and $N \subseteq \bigcap_{n\in\omega} W_n$. Clearly, the identity mapping of $G/N$ onto $G_*/N$ is continuous. Hence, by Lemma 2.5, the identity mapping of $(G/N)_{sr}$ onto $G_*/N$ is also continuous. Since $G_*/N$ is submetrizable, the space $(G/N)_{sr}$ is submetrizable and hence Hausdorff. So by Theorem 1.1, the space $(G/N)_{sr}$ is regular.

Let $f$ be the quotient mapping of $G$ onto $G/N$. Then $f(B)$ is bounded in $G/N$ and in $(G/N)_{sr}$. Denote by $C$ the closure of $f(B)$ in $(G/N)_{sr}$. Applying Corollary 2.2, we conclude that $C$ is compact. This implies that the set $f(B)$ is strongly bounded in $(G/N)_{sr}$. By Lemma 2.4, $f(B)$ is strongly bounded in $G/N$.

Put $O_n = f(x_n W_n)$, for every $n \in \omega$. Since the mapping $f$ is open, $O_n$ is open in $G/N$ and $O_n \cap f(B) \neq \emptyset$ for each $n \in \omega$. As $f(B)$ is strongly bounded in $G/N$, we conclude that

$$\bigcap_{F\in\mathcal{F}} \overline{\bigcup_{n\in F} O_n} \neq \emptyset.$$

Take a point $y$ in the above intersection and choose $x \in G$ such that $f(x) = y$. We claim that $x \in \bigcap_{F\in\mathcal{F}} \overline{\bigcup_{n\in F} U_n}$. Indeed, let $F$ be an arbitrary element of $\mathcal{F}$. Since the mapping $f$ is open and continuous, we have that

$$x \in f^{-1}\left(\overline{\bigcup_{n\in F} O_n}\right) = \overline{f^{-1}\big(\bigcup_{n\in F} O_n\big)} = \overline{\bigcup_{n\in F} f^{-1}(O_n)}$$
$$= \overline{\bigcup_{n\in F} x_n W_n N} \subseteq \overline{\bigcup_{n\in F} x_n W_n^2} \subseteq \overline{\bigcup_{n\in F} U_n}.$$

It follows that $x \in \bigcap_{F\in\mathcal{F}} \overline{\bigcup_{n\in F} U_n} \neq \emptyset$, so the family $\{U_n : n \in \omega\}$ satisfies $(*)$ of Definition 2.3. Thus $B$ is strongly bounded in $G$. □

**Remark 3.8.** The argument in the proof of Theorem 3.7 implies the following: *If the regular reflection, $Reg(G)$, of a paratopological group $G$ is 2-oscillating and satisfies $Hs(Reg(G)) \leq \omega$, then every bounded subset of $G$ is strongly bounded.*

By Lemma 3.6, the latter result is a more general and symmetric form of Theorem 3.7.

According to [14, Theorem 2.2], every regular totally $\omega$-narrow paratopological group $G$ satisfies $Ir(G) \leq \omega$. The inequality $Ir(G) \leq \omega$ means that for every neighborhood $U$ of the neutral element $e$ in $G$, one can find an open neighborhood $V$ of $e$ and a countable family $\gamma$ of neighborhoods of $e$ such that $\bigcap_{W\in\gamma} VW^{-1} \subseteq U$. Since the inequality $Hs(G) \leq Ir(G)$ holds for every regular paratopological group [23, Proposition 3.5], we see that the class of Hausdorff (even regular) paratopological groups with countable Hausdorff number is strictly wider than the class of regular totally $\omega$-narrow paratopological groups. Hence the next result complements Theorem 2.7.

**Corollary 3.9.** *Every bounded subset of a Hausdorff commutative paratopological group $G$ with $Hs(G) \leq \omega$ is strongly bounded.*

**Proof.** It is clear that every commutative paratopological group is 2-oscillating. Hence the conclusion follows from Theorem 3.7. □

Since the product of a family of strongly bounded subsets is strongly bounded in the product of spaces [22, Theorem 2.6], the following result is immediate from Corollary 3.9.

**Corollary 3.10.** *Let $B_i$ be a bounded subset of a Hausdorff commutative paratopological group $G_i$ satisfying $Hs(G_i) \leq \omega$, where $i \in I$. Then the set $\prod_{i\in I} B_i$ is bounded in $\prod_{i\in I} G_i$.*

We recall that a paratopological group $G$ is *saturated* if the set $U^{-1}$ has a non-empty interior, for each neighborhood $U$ of the neutral element in $G$.

**Corollary 3.11.** *If a saturated paratopological group $G$ satisfies $Hs(Reg(G)) \leq \omega$, then all bounded subsets of $G$ are strongly bounded.*

**Proof.** By [3, Proposition 3], every saturated paratopological group is 2-oscillating. It remains to apply Theorem 3.7. □

A space $X$ is said to be *weakly Lindelöf* if every open covering of $X$ contains a countable subfamily whose union is dense in $X$. Every space of countable cellularity as well as a space with a dense Lindelöf subspace is weakly Lindelöf.

The next result is a special case of Theorem 3.7.

**Theorem 3.12.** *Let $G$ be a weakly Lindelöf paratopological group. If the group $Reg(G)$ is saturated, then every bounded subset of $G$ is strongly bounded.*

**Proof.** Suppose that the regular reflection of $G$, say, $H = Reg(G)$, is saturated. Then the group $H$ is 2-oscillating by [3, Proposition 3]. Since $H$ is a continuous homomorphic image of $G$, the space $H$ is weakly Lindelöf. Therefore, combining [14, Theorem 2.10] and [23, Proposition 3.5], we see that the Hausdorff number of the group $H$ is countable. Then Theorem 3.7 implies that every bounded subset of $H$ is strongly bounded. Finally it remains to apply Proposition 3.2 to conclude that every bounded subset of $G$ is strongly bounded. □

It turns out that the operation of regular reflection preserves the class of saturated paratopological groups:

**Lemma 3.13.** *If a paratopological group $G$ is saturated, so is $Reg(G)$.*

**Proof.** It is clear that a continuous open homomorphic image of a saturated paratopological group is saturated. Since the canonical homomorphism $\varphi_{G,2}\colon G \to T_2(G)$ is continuous and open by [27, Proposition 2.5], we conclude that the Hausdorff reflection, $T_2(G)$, of the saturated group $G$ is saturated.

According to Proposition 3.3, the group $Reg(G)$ is topologically isomorphic to the semiregularization of $T_2(G)$. Therefore, to finish the proof, it suffices to verify that the semiregularization of a saturated paratopological group, say, $H$ is saturated as well. Let $U$ be an arbitrary open neighborhood of the neutral element $e$ in $H$. Take an open neighborhood $V$ of $e$ in $H$ such that $V^2 \subseteq U$ and a non-empty open set $W$ in $G$ satisfying $W \subseteq V^{-1}$. Then $\overline{W} \subseteq WV^{-1} \subseteq V^{-1}V^{-1} \subseteq U^{-1}$. Therefore,

$$\operatorname{Int}\overline{W} \subseteq \overline{W} \subseteq U^{-1} \subseteq (\operatorname{Int}\overline{U})^{-1}.$$

Since the sets of the form $\operatorname{Int}\overline{O}$ constitute a base for the group $H_{sr}$, where $O$ runs through the family of non-empty open sets in $H$, we conclude that the group $H_{sr}$ is saturated. □

Let us note that the implication in Lemma 3.13 cannot be inverted. Furthermore, there exists a non-saturated paratopological group $G$ such that the regular reflection of $G$ is the trivial one-element group. Indeed, consider the additive group $\mathbb{R}$ endowed with the topology $\tau$ whose base consists of the sets $[x, \infty)$, with $x \in \mathbb{R}$. Then $G = (\mathbb{R}, \tau)$ is a paratopological group and the set $-[0, \infty) = (-\infty, 0]$ has the empty interior, i.e. $G$ is not saturated. It is easy to verify that the regular reflection of $G$ is the trivial group (and, hence, is saturated).

It follows from Theorem 3.12 and Lemma 3.13 that every bounded subset of a saturated, weakly Lindelöf paratopological group is strongly bounded. Hence the following result is valid:

**Corollary 3.14.** *Let $B_i$ be a bounded subset of a saturated, weakly Lindelöf paratopological group $G_i$, where $i \in I$. Then the set $\prod_{i\in I} B_i$ is bounded in $\prod_{i\in I} G_i$.*

Since every precompact paratopological group is saturated [11, Proposition 2.1] and has countable cellularity [3, Corollary 3], the following corollary is immediate from Theorem 3.12 and Lemma 3.13:

**Corollary 3.15.** *Every bounded subset of a precompact paratopological group $G$ is strongly bounded.*

**Corollary 3.16.** *Let $B_i$ be a bounded subset of a precompact paratopological group $G_i$, for each $i \in I$. Then $\prod_{i\in I} B_i$ is bounded in $\prod_{i\in I} G_i$.*

The following fact is of a pure topological character. It will be applied to paratopological groups in Theorem 3.18.

**Proposition 3.17.** *The closure of a bounded subset $B$ of a regular Lindelöf space $X$ is compact and, hence, $B$ is strongly bounded in $X$.*

**Proof.** Denote by $K$ the closure of $B$ in $X$. Then $K$ is also bounded in $X$. The space $X$ is normal since it is regular and Lindelöf. Hence the closed set $K$ is $C$-embedded in $X$. Therefore $K$ is pseudocompact as a $C$-embedded bounded subset of a Tychonoff space. The subspace $K$ of $X$ being closed is also normal. We conclude that $K$ is countably compact [8, Theorem 3.10.21]. Hence $K$ is compact as a Lindelöf countably compact space and $B$ is strongly bounded in $X$. □

Another version of Theorem 2.7 is given below.

**Theorem 3.18.** *Let $G$ be a paratopological group whose regular reflection is Lindelöf. Then every bounded subset of $G$ is strongly bounded in $G$.*

**Proof.** According to Proposition 3.2, a subset $B$ of $G$ is (strongly) bounded in $G$ if and only if $\varphi_{G,r}(B)$ is (strongly) bounded in $Reg(G)$, where $\varphi_{G,r}$ is the canonical homomorphism of $G$ onto $Reg(G)$. Therefore, we can assume that $Reg(G) = G$, i.e. that $G$ is regular. The required conclusion now follows from Proposition 3.17. □

Since the property of being strongly bounded is productive, Theorem 3.18 implies the following:

**Corollary 3.19.** *If $B_i$ is a bounded subset of a Lindelöf paratopological group $G_i$, where $i \in I$, then $\prod_{i\in I} B_i$ is bounded in $\prod_{i\in I} G_i$.*

One cannot strengthen 'strongly bounded' to 'compact' in the conclusion of Theorem 3.18 since there exists a Hausdorff, first countable, feebly compact, non-compact paratopological group (see [12, Example 3]).

## 4. Some discussions

In this short section we comment on how one can modify conditions in Theorems 3.7, 3.12, and 3.18 and then compare the generality of the modified results with the original ones.

The paratopological group $G$ in Theorem 3.7 is assumed to be 2-oscillating and to satisfy the inequality $Hs(Reg(G)) \leq \omega$. The first of these conditions refers to the group $G$ itself, while the second condition concerns to the regular reflection $Reg(G)$. It is natural to ask, therefore, whether we really gain in generality referring to $Reg(G)$ in place of $G$ in Theorem 3.7.

First we comment on the condition $Hs(Reg(G)) \leq \omega$ that appears in Theorem 3.7. In the following two results we establish clear relations between the Hausdorff numbers of the paratopological groups $G$ (when $Hs(G)$ is defined), $T_2(G)$, and $Reg(G)$.

**Lemma 4.1.** *If $G$ is a paratopological group and the Hausdorff number of $G$ is defined, i.e. $G$ has property (wH), then $Hs(G) = Hs(T_2(G))$.*

**Proof.** Let $P = \bigcap \mathcal{N}(e)$, where $\mathcal{N}(e)$ is the family of open neighborhoods of the identity $e$ in $G$. By [27, Theorem 3.1], $N = P \cap P^{-1}$ is the kernel of the canonical homomorphism $\varphi_{G,0}\colon G \to T_0(G)$. If $G$ has property (wH), then $P = \bigcap_{U\in\mathcal{N}(e)} \overline{U}$. The set on the right hand part of the equality is the kernel of the canonical homomorphism $\varphi_{G,2}\colon G \to T_2(G)$ and, hence, a closed subgroup of $G$ [28, Theorem 2.1]. We conclude that $P$ is symmetric and that $N = P$. In turn, the two equalities involving $P$ imply that the kernels of the homomorphisms $\varphi_{G,0}$ and $\varphi_{G,2}$ coincide. Since the homomorphisms $\varphi_{G,0}$ and $\varphi_{G,2}$ are open by [27, Proposition 2.5], it follows that the groups $T_0(G)$ and $T_2(G)$ are topologically isomorphic. More precisely, there exists a topological isomorphism $f: T_0(G) \to T_2(G)$ satisfying $\varphi_{G,2} = f \circ \varphi_{G,0}$.

We now apply [27, Theorem 3.1] according to which every open set $U$ in $G$ satisfies $U = \varphi_{G,0}^{-1}\varphi_{G,0}(U)$. Hence the same equality is valid for $\varphi_{G,2}$ in place of $\varphi_{G,0}$. In other words, the correspondence $U \mapsto \varphi_{G,2}(U)$ is a bijection between the topologies of $G$ and $T_2(G)$. This readily yields the required equality $Hs(G) = Hs(T_2(G))$. □

Lemma 4.1 implies, in particular, that one can define the Hausdorff number of an *arbitrary* paratopological group $G$ by letting $Hs(G) = Hs(T_2(G))$. This definition becomes even more natural if we recall [28, Corollary 2.3]: If $U$ and $V$ are disjoint open sets in $G$, then $\varphi_{G,2}(U)$ and $\varphi_{G,2}(V)$ are disjoint open sets in $T_2(G)$.

**Lemma 4.2.** *Every paratopological group $K$ with property (wH) satisfies the inequality $Hs(K_{sr}) \le Hs(K)$. Therefore, $Hs(Reg(G)) \le Hs(T_2(G))$, for each paratopological group $G$.*

**Proof.** Since the group $K$ has property (wH) and the semiregularization $K_{sr}$ of $K$ is a $T_3$-space, the Hausdorff number of both groups $K$ and $K_{sr}$ is well defined. Let $U$ be a neighborhood of the neutral element $e$ in the group $K$ satisfying $Hs(K) = \kappa$. There exists a family $\gamma$ of open neighborhoods of $e$ in $K$ such that $\bigcap_{V\in\gamma} VV^{-1} \subseteq U$ and $|\gamma| \le \kappa$. For every $V \in \gamma$, let $O_V = \mathrm{Int}\,\overline{V}$. Then $\lambda = \{O_V : V \in \gamma\}$ is a family of open neighborhoods of $e$ in $K_{sr}$ and $|\lambda| \le |\gamma| \le \kappa$.

We claim that $\bigcap_{V\in\gamma} O_V O_V^{-1} \subseteq U$. Indeed, take an arbitrary point $x \in K \setminus U$. It follows from our choice of $\gamma$ that there exists $V \in \gamma$ such that $V \cap xV = \emptyset$. Since the sets $V$ and $xV$ are open in $K$ and dense in $O_V$ and $xO_V$, respectively, we see that $O_V \cap xO_V = \emptyset$ or, equivalently, $x \notin O_V O_V^{-1}$. Hence $\bigcap_{V\in\gamma} O_V O_V^{-1} \subseteq U$.

It is clear that the last inequality of the lemma follows from the first one and Proposition 3.3. □

It turns out that the difference between $Hs(G)$ and $Hs(Reg(G))$ can be arbitrarily big, even for a Hausdorff paratopological group $G$. Indeed, a simple calculation shows that the feebly compact Hausdorff paratopological group $G$ in [20, Theorem 1] satisfies $Hs(G) = \aleph_1$, while $G_{sr} = Reg(G)$ is a topological group and, hence, $Hs(Reg(G)) = 1$. Given an arbitrary uncountable cardinal $\kappa$, a similar construction can be applied to produce a feebly compact Hausdorff paratopological group $H$ satisfying $Hs(Reg(H)) = 1$ and $Hs(H) = \kappa$. This observation and Lemma 4.2 explain why we use the Hausdorff number of the regular reflection of $G$ instead of $Hs(G)$ or $Hs(T_2(G))$ in Theorem 3.7.

Finally we show that the Hausdorff numbers of the groups $G_{sr}$ and $Reg(G)$ always coincide. This requires a lemma.

**Lemma 4.3.** *Let $G$ be a paratopological group satisfying the $T_3$ separation axiom. Then $Hs(G) = Hs(T_0(G))$.*

**Proof.** Let $\mathcal{N}(e)$ be the family of open neighborhoods of the neutral element $e$ in $G$ and $N = \bigcap_{U\in\mathcal{N}(e)}(U \cap U^{-1})$. According to [27, Theorem 3.1], $T_0(G) \cong G/N$. Let $f$ be the quotient homomorphism of $G$ to $G/N$. Fix an element $U \in \mathcal{N}(e)$ and choose $V \in \mathcal{N}(e)$ such that $V^2 \subseteq U$. We can find a family $\gamma$ of open neighborhoods of the neutral element $\bar{e}$ in $G/N$ such that $\bigcap_{O\in\gamma} OO^{-1} \subseteq f(V)$ and $|\gamma| \le Hs(G/N)$. For every $O \in \gamma$, there exists $W_O \in \mathcal{N}(e)$ such that $f(W_O) \subseteq O$. Then $\bigcap_{O\in\gamma} f(W_O W_O^{-1}) \subseteq f(V)$ and $\bigcap_{O\in\gamma} f^{-1}f(W_O W_O^{-1}) \subseteq f^{-1}f(V) = VN$, whence

$$\bigcap_{O\in\gamma} W_O W_O^{-1} \subseteq \bigcap_{O\in\gamma} f^{-1}f(W_O W_O^{-1}) \subseteq VN \subseteq V^2 \subseteq U.$$

This proves that $Hs(G) \le Hs(T_0(G))$.

It remains to show that $Hs(T_0(G)) \le Hs(G)$. Take an arbitrary neighborhood $O$ of $\bar{e}$ in $G/N$. We can assume that $O = f(U)$, for some $U \in \mathcal{N}(e)$. There exists a family $\{V_i : i \in I\} \subseteq \mathcal{N}(e)$ such that $|I| \le Hs(G)$ and $\bigcap_{i\in I} V_i V_i^{-1} \subseteq U$. For every $i \in I$, take $W_i \in \mathcal{N}(e)$ such that $W_i^2 \subseteq V_i$. We claim that $\bigcap_{i\in I} f(W_i)f(W_i)^{-1} \subseteq O$.

Indeed, since $N \subseteq W_i^{-1}$ and $W_i \subseteq W_i^2 \subseteq V_i$ for each $i \in I$, it follows that

$$\bigcap_{i\in I} f^{-1}f(W_i W_i^{-1}) = \bigcap_{i\in I} W_i W_i^{-1} N \subseteq \bigcap_{i\in I} V_i V_i^{-1} \subseteq U.$$

We conclude that $\bigcap_{i\in I} f(W_i)f(W_i)^{-1} \subseteq f(U) = O$. Therefore, $Hs(T_0(G)) \le Hs(G)$. This finishes the proof. □

**Corollary 4.4.** *If $G$ is a paratopological group, then $Hs(G_{sr}) = Hs(Reg(G))$.*

**Proof.** Put $H = G_{sr}$. Then $H$ is a paratopological group satisfying the $T_3$ separation axiom, so [27, Proposition 3.7] implies that $Reg(G) = T_0(H)$. Applying Lemma 4.3, we obtain the required conclusion. □

## 5. Open problems and comments

One of the main open problems about bounded subsets of paratopological groups is the following one:

**Problem 5.1.** Let $B$ be a bounded subset of a paratopological group $G$.

(a) Is $B$ strongly bounded in $G$?
(b) Is $B$ strongly bounded in $G$ provided $B$ is countably compact?

It is clear that Theorems 2.7, 3.7, 3.12, 3.18 and Corollaries 3.9, 3.11, 3.15 answer item (a) of Problem 5.1 affirmatively in wide subclasses of paratopological groups.

Problems 5.2, 5.3, 5.6 and 5.7 below are special cases of Problem 5.1. In the next one we ask whether the condition "2-oscillating" imposed on the group $G$ in Theorem 3.7 can be dropped:

**Problem 5.2.** Suppose that a Hausdorff paratopological group $G$ satisfies $Hs(G) \le \omega$. Is every bounded subset of $G$ strongly bounded?

A slightly weaker form of Problem 5.2 is given below:

**Problem 5.3.** Let $B_i$ be a bounded subset of a Hausdorff paratopological group $G_i$ satisfying $Hs(G_i) \le \omega$, where $i \in I$. Is the set $\prod_{i\in I} B_i$ bounded in $\prod_{i\in I} G_i$?

It follows from [17, Proposition 3.2.14] that a Hausdorff paratopological group $G$ of countable $\pi$-character satisfies $Hs(G) \le \omega$. So it is natural to ask, after Problem 5.2, whether every bounded subset of a Hausdorff paratopological group of countable $\pi$-character is strongly bounded. The question is also motivated by [14, Corollary 26] according to which every bounded subset of a regular paratopological group with countable $\pi$-character is metrizable. We answer the above question in the affirmative and improve upon Corollary 26 of [14]. First we define the notion of a countably Hausdorff group introduced in [5] and then prove Lemma 5.4 on which the proof of Proposition 5.5 is based.

According to [5], a paratopological group $H$ is *countably Hausdorff* if there exists a countable family $\{W_n : n \in \omega\}$ of open neighborhoods of the identity in $H$ such that for every pair $x, y$ of distinct elements of $H$, one can find $n \in \omega$ satisfying $xW_n \cap W_n y = \emptyset$.

**Lemma 5.4.** *([15]) Every Hausdorff paratopological group $H$ with countable $\pi$-character is countably Hausdorff.*

**Proof.** Let $\{U_n : n \in \omega\}$ be a local $\pi$-base at the identity $e$ in $H$. Put $W_n = U_n U_n^{-1} \cap U_n^{-1} U_n$ for every $n \in \omega$. Now, take two distinct elements $x, y \in H$. Since $H$ is Hausdorff, there exists an open neighborhood $V$ of $e$ in $H$ such that $VxV \cap VyV = \emptyset$. So $xVV^{-1} \cap V^{-1}Vy = \emptyset$. We can find $n \in \omega$ with $U_n \subseteq V$. Therefore, $xW_n \cap W_n y \subseteq xU_n U_n^{-1} \cap U_n^{-1} U_n y = \emptyset$. □

**Proposition 5.5.** *Every bounded subset of a Hausdorff paratopological group $G$ of countable $\pi$-character is strongly bounded. In addition, if $G$ is regular, then every closed bounded subset of $G$ is compact and metrizable.*

**Proof.** By Lemma 2.4, it suffices to show that every bounded subset of the semiregularization $G_{sr}$ of $G$ is strongly bounded. First we claim that the group $G_{sr}$ has countable $\pi$-character. Indeed, let $\{U_n : n \in \omega\}$ be a countable $\pi$-base at the identity $e$ of the group $G$. For every $n \in \omega$, let $V_n$ be the interior of the closure of $U_n$ in $G$. It is clear that each $V_n$ is a nonempty open set in $G_{sr}$. Let us show that the family $\{V_n : n \in \omega\}$ is a $\pi$-base at $e$ in $G_{sr}$. Take an open neighborhood $W$ of $e$ in $G_{sr}$. There exists an open neighborhood $U$ of $e$ in $G$ such that $\mathrm{Int}_G \overline{U} \subseteq W$. By our assumption, there exists $n \in \omega$ such that $U_n \subseteq U$. Then $V_n = \mathrm{Int}_G \overline{U_n} \subseteq \mathrm{Int}_G \overline{U} \subseteq W$, whence our claim follows.

By Lemma 5.4, $G_{sr}$ is countably Hausdorff. Then we apply a recent result proved by Banakh and Ravsky in [5, Theorem 5]: Every countably Hausdorff paratopological group is submetrizable. Therefore, $H = G_{sr}$ is a regular submetrizable paratopological group. Further, every regular paratopological group is Tychonoff, by [5, Corollary 5]. Every Tychonoff submetrizable space is Dieudonné complete, so the closure of every bounded subset of $H$ is compact. Hence every bounded subset of $H$ is strongly bounded. Notice that by item (a) of Corollary 2.2, the bounded subsets of $H$ are metrizable. □

We note, in connection with Proposition 5.5, that a closed bounded subset of a Hausdorff, first countable, $\omega$-narrow paratopological group may fail to be compact. Indeed, there exists a feebly compact, $\omega$-narrow, first countable, Hausdorff paratopological group which is not compact [13, Example 3].

Since precompact paratopological groups are saturated, the following problem arises in an attempt to generalize Corollary 3.15.

**Problem 5.6.** Let $G$ be a saturated, $\omega$-narrow paratopological group. Is every bounded subset of $G$ strongly bounded?

In fact, we do not know whether every saturated, $\omega$-narrow, regular paratopological group $G$ satisfies $Hs(G) \leq \omega$. However, there exists a precompact (hence saturated and $\omega$-narrow) Hausdorff paratopological group with uncountable Hausdorff number (see [14, Example 18]).

Given a paratopological group $G$, we denote by $Sm(G)$ the *symmetry number* of $G$ defined as the minimum cardinal number $\kappa$ such that for every neighborhood $U$ of the neutral element $e$ in $G$, there exists a family $\gamma$ of open neighborhoods of $e$ in $G$ such that $\bigcap \gamma \subseteq U^{-1}$ and $|\gamma| \leq \kappa$. It is known that $Sm(G) \leq Hs(G)$, for each Hausdorff paratopological group $G$ (see [16]). Slightly modifying [16, Example 2.29] we can construct, for every uncountable cardinal $\kappa$, a commutative totally $\omega$-narrow Hausdorff paratopological group $H$ with $Sm(H) = \kappa$ and $Hs(H_{sr}) = 1$.

One can try to generalize Corollary 3.9 by replacing the Hausdorff number with the symmetry number:

**Problem 5.7.** Is every bounded subset of a Hausdorff commutative paratopological group $G$ with $Sm(G) \leq \omega$ strongly bounded?

Very recently the second listed author complemented Theorem 3.12 as follows (see [30]):

**Theorem 5.8.** *Every bounded subset of a weakly Lindelöf $\omega$-balanced paratopological group $G$ is strongly bounded.*

The proof of Theorem 5.8 leans on the following facts. First, one can assume without loss of generality that the group $G$ is regular (see Lemma 2.4). Second, every regular paratopological group is Tychonoff [5, Corollary 5]. Third, the Dieudonné completion and Hewitt–Nachbin completion of the Tychonoff paratopological group $G$ coincide and both contain $G$ as a dense subgroup [30]. Let $H$ be the Dieudonné completion of $G$. Then $G$ is a dense $C$-embedded subspace of $H$, so $G$ meets every nonempty $G_\delta$-set in $H$. Hence Corollary 3.6 of [21] implies that every bounded subset of $G$ is strongly bounded.

We do not know, however, whether '$\omega$-balanced' can be dropped in Theorem 5.8:

**Problem 5.9.** Is every bounded subset of a weakly Lindelöf paratopological group strongly bounded?

There exist several notions close to boundedness. A subset $B$ of a space $X$ is called *relatively pseudocompact* in $X$ if every infinite family of open sets in $X$ each of which meets $B$ has an accumulation point in $B$. It is also said that $B$ is *$C$-compact* in a Tychonoff space $X$ if the image $f(B)$ is compact, for each continuous real-valued function $f$ on $X$. The following implications are almost immediate (see [10, Section 2]):

$$\text{pseudocompact} \Rightarrow \text{relatively pseudocompact} \Rightarrow C\text{-compact} \Rightarrow \text{bounded}.$$

According to [10, Corollary 3.11], every $C$-compact subset of a topological group is relatively pseudocompact, while $C$-compactness is productive in topological groups by [9, Corollary 3]. It would be interesting to find out whether these results remain valid in the class of paratopological groups:

**Problem 5.10.** Does $C$-compactness imply relative pseudocompactness in paratopological groups?

**Problem 5.11.** Is $C$-compactness productive in Tychonoff paratopological groups?

The authors have recently found in [18] several classes of paratopological groups in which the answer to both Problems 5.10 and 5.11 is affirmative.

## References


[1] A.V. Arhangel'skii, On a theorem of W.W. Comfort and K.A. Ross, Comment. Math. Univ. Carol. 40 (1999) 133–151.
[2] A.V. Arhangel'skii, M.G. Tkachenko, Topological Groups and Related Structures, Atlantis Series in Mathematics, vol. 1, Atlantis Press/World Scientific, Paris–Amsterdam, 2008.
[3] T. Banakh, O. Ravsky, Oscillator topologies on a paratopological group and related number invariants, in: Algebraical Structures and Their Applications, Inst. Mat. NANU, Kyiv, 2002, pp. 140–153.
[4] T. Banakh, A. Ravsky, The regularity of quotient paratopological groups, arXiv:1003.5409v2 [math.GR], 2014.
[5] T. Banakh, A. Ravsky, A characterization of completely regular spaces with applications to paratopological groups, arXiv:1410.1504v7 [math.GN], January 10, 2015.
[6] A.C. Chigogidze, On $\kappa$-metrizable spaces, Russ. Math. Surv. 37 (1982) 209–210, Russian original in: Usp. Mat. Nauk 37 (1982) 241–242.
[7] W.W. Comfort, K.A. Ross, Pseudocompactness and uniform continuity in topological groups, Pac. J. Math. 16 (1966) 483–496.
[8] R. Engelking, General Topology, Heldermann Verlag, Berlin, 1989.
[9] S. Hernández, M. Sanchis, $G_\delta$-open functionally bounded subsets in topological groups, Topol. Appl. 53 (1993) 289–299.
[10] S. Hernández, M. Sanchis, M. Tkachenko, Bounded sets in spaces and topological groups, Topol. Appl. 101 (2000) 21–43.
[11] A. Ravsky, Paratopological groups II, Mat. Stud. 17 (1) (2002) 93–101.
[12] A. Ravsky, Pseudocompact paratopological groups, arXiv:1003.5343v5 [math.GR], 2013.
[13] A. Ravsky, Pseudocompact paratopological groups that are topological, arXiv:1003.5343v3 [math.GR], 2012.
[14] I. Sánchez, Cardinal invariants of paratopological groups, Topol. Algebra Appl. (2013) 37–45, http://dx.doi.org/10.2478/taa-2013-0005.

[15] I. Sánchez, Dense subgroups of paratopological groups, Preprint, 2015.
[16] I. Sánchez, Subgroups of products of paratopological groups, Topol. Appl. 163 (2014) 160–173.
[17] I. Sánchez, Paratopological groups, PhD thesis, Dept. of Math., Universidad Autónoma Metropolitana, Mexico City, 2013 (in Spanish).
[18] I. Sánchez, M. Tkachenko, $C$-compactness and $r$-pseudocompactness in paratopological groups, Topol. Appl. (2015), in press.
[19] M. Sanchis, Problems on bounded sets, Quest. Answ. Gen. Topol. 28 (2010) 65–79.
[20] M. Sanchis, M.G. Tkachenko, Feebly compact paratopological groups and real-valued functions, Monatshefte Math. 168 (2012) 579–597.
[21] M. Sanchis, M.G. Tkachenko, Completions of paratopological groups and bounded sets, Preprint, 2015.
[22] M. Tkachenko, Compactness type properties in topological groups, Czechoslov. Math. J. 38 (1988) 324–341.
[23] M.G. Tkachenko, Embedding paratopological groups into topological products, Topol. Appl. 156 (2009) 1298–1305.
[24] M.G. Tkachenko, Generalization of a theorem of Comfort and Ross. II, Ukr. Math. J. 41 (1990) 802–806, Russian original in: Ukr. Mat. Zh. 41 (1989) 939–944.
[25] M.G. Tkachenko, Group reflection and precompact paratopological groups, Topol. Algebra Appl. 1 (2013) 22–30, http://dx.doi.org/10.2478/taa-2013-0003.
[26] M. Tkachenko, Paratopological and semitopological groups vs topological groups, in: J.P. Hart, J. van Mill, P. Simon (Eds.), Recent Progress in General Topology III, Atlantis Press, 2014, pp. 803–859.
[27] M. Tkachenko, Axioms of separation in semitopological groups and related functors, Topol. Appl. 161 (2013) 364–376.
[28] M. Tkachenko, Axioms of separation in paratopological groups and reflection functors, Topol. Appl. 179 (2015) 200–214.
[29] M. Tkachenko, Applications of the reflection functors in paratopological groups, Topol. Appl. (2015), in press.
[30] M. Tkachenko, The Hewitt–Nachbin completion of paratopological groups, Preprint, 2015.
[31] F.J. Trigos-Arrieta, Products of locally pseudocompact topological groups, in: Papers on General Topology and Applications, Brookville, NY, 1990, in: Ann. N.Y. Acad. Sci., vol. 659, 1992, pp. 182–187, New York Acad. Sci., New York.
[32] V.V. Uspenskij, Topological groups and Dugundji compacta, Math. USSR Sb. 67 (1990) 555–580, Russian original in: Mat. Sb. 180 (1989) 1092–1118.
[33] L.-H. Xie, S. Lin, M. Tkachenko, Factorization properties of paratopological groups, Topol. Appl. 160 (14) (2013) 1902–1917.